[redacted]

NACA TM 1272

# NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

## TECHNICAL MEMORANDUM 1272

# CRITICAL VELOCITIES OF ULTRACENTRIFUGES

By V. I. Sokolov

Translation

"Criticheskie Skorosti Supertsentrifug." Zhurnal Tekhnicheskoi Fiziki. (U.S.S.R.). Vol. XVI, no. 4, 1946.

Washington
March 1951

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1272

# CRITICAL VELOCITIES OF ULTRACENTRIFUGES*

By V. I. Sokolov

In chemical industry, it is often necessary to deal with the phenomenon of instability of operation of ultracentrifuges that lead, in a number of cases, to breakdown, accidents, and early wear of the instrument. In these cases, understanding of the phenomenon that may help to remedy the condition is not always found.

The characteristics of the passage through the critical velocities of ultracentrifuges are discussed and the corresponding practical conclusions are presented in this paper.

The Euler differential equations of rotation of a solid body are applied.

The following constructions are made: the fixed axes $\xi$, $\eta$, and $\vartheta$ with the origin at the point O (fig. 1); the axis of symmetry of the centrifuge rotor $z$; the nodal line $I$ perpendicular to the plane of $\vartheta$, $z$; and finally, the axis $k$ perpendicular to the axes $z$, $I$.

By applying the Euler equations of the rotation of a solid body and according to the notation of Nikolai (reference 1)

$$\left.\begin{aligned}
\frac{d\sigma_1}{dt} + q_1\,\sigma_z - r_1\,\sigma_k &= L_I \\
\frac{d\sigma_k}{dt} + r_1\,\sigma_1 - p_1\,\sigma_z &= L_k \\
\frac{d\sigma_z}{dt} + p_1\,\sigma_k - q_1\,\sigma_1 &= L_z
\end{aligned}\right\} \tag{1}$$

The rotation of the centrifuge rotor may be considered as a regular precession of the symmetrical gyroscope. The angular velocity $\omega$ of the rotor is therefore the angular velocity of this precession and the

*"Criticheskie Skorosti Supertsentrifug." Zhurnal Tekhnicheskoi Fiziki. (U.S.S.R.). Vol. XVI, no. 4, 1946, pp. 463-468.

axis $\vartheta$ is the axis of precession. By laying off the angular velocity of the drum, or more accurately, the trihedron formed by the axes I, k, and z that are assumed constant on the axis $\vartheta$, and by projecting it on the axes I, k, and z, the following equations are obtained:

$$\left.\begin{aligned} p_1 &= \dot{\beta} \\ q_1 &= \omega \sin \beta \\ r_1 &= \left| \omega + \dot{\alpha} \right| \cos \beta \end{aligned}\right\} \tag{2}$$

The velocity $\dot{\alpha}$ is the result of the action of the Coriolis forces and, by comparison with $\omega$, is a small magnitude. For simplification of the solution, it is assumed that

$$r_1 = \omega \cos \dot{\beta} \tag{2a}$$

In other words, the assumption, often used in the theory of vibrations, is made that the principal axis of the rotor z always remains in the plane including the vertical (the initial position of the axis of the shaft) and rotates with the same constant angular velocity $\omega$ with which the drum rotates.

The center of inertia of the rotor is assumed to lie on its axes of symmetry $O_1$ and $O_2$. The axes I, k, and z will then be the principal axes of inertia of the rotor at the point $O_1$ because, from the symmetry in regard to the axis z, all straight lines drawn perpendicular to this axis through the point $O_1$ are principal axes of inertia. From this assumption, it follows that the moments $\sigma_I, \sigma_k, \sigma_z$ are

$$\sigma_1 = Ap$$

$$\sigma_k = Aq$$

$$\sigma_z = Cr$$

where A and C are the equatorial (at point $O_1$) and polar moments of inertia.

From equations (2)

$$\left.\begin{aligned} \sigma_1 &= A\dot{\beta} \\ \sigma_k &= A\omega \sin\beta \\ \sigma_z &= C\omega \cos\beta \end{aligned}\right\} \qquad (3)$$

By linearizing, the following equations are obtained:

$$\left.\begin{aligned} p_1 &= \beta \\ \sigma_1 &= A\dot{\beta} \\ q_1 &= \omega\beta \\ \sigma_k &= A\omega\beta \\ r_1 &= \omega \\ \sigma_z &= C\omega \end{aligned}\right\} \qquad (4)$$

In equations (1), $L_k = L_z = 0$ but the moment of the external forces applied to the rotor relative to the axis I is determined by the expression

$$L_1 = -\delta_{11} h^2 \sin\beta + Gh_1 \sin\beta \qquad (5)$$

where $\delta_{11}$ is the force applied at the end of the rotor spindle that produces unit rotation in the case where the spindle is immovably fixed, and G is the weight of the rotor; h and $h_1$ are shown in figure 1.

By neglecting the gravitational force and by replacing $\sin\beta$ by $\beta$

$$L_I = -\delta_{11} h^2 \beta \qquad (6)$$

By substituting in the Euler equation the values of the magnitudes that are used, the following equation is obtained:

$$\omega^2 \left| C - A \right| \beta = -\delta_{11} h^2 \beta \qquad (7)$$

Because the nonuniform distribution of the mass of the rotor results in a deviation of the principal axis of inertia from the vertical by the angle $\beta_0$, equation (7) must be written as

$$\omega^2 \left| C - A \right| \beta + \beta_0 \Big| = - \delta_{11} h^2 \beta \tag{8}$$

where

$$\beta = \beta_0 \frac{\omega^2}{\frac{-\delta_{11} h^2}{C - A} - \omega^2} \tag{9}$$

Hence the critical velocity is

$$\omega_{cr} = \sqrt{\frac{\delta_{11} h^2}{A - C}} \tag{10}$$

In order to complete the solution and obtain a computation formula, $\delta_{11}$ is determined.

In the corresponding constructions of the usual type of centrifuges, the spindle is assumed to be of conical shape. In determining $\delta_{11}$, the spindle may be considered as a beam with the large cross section built in. The fixing of the beam corresponds to the joining to the spindle with the centrifuge rotor (fig. 2).

On the basis of the Mohr formula, the displacement of the end of the beam $\Delta_{11}$ may be determined by the equation

$$\Delta_{11} = \int_{a}^{a+l} \frac{M^2}{EI} dx \tag{11}$$

where $M$ is the bending moment, $E$ is the modulus of elasticity, and $I$ is the moment of inertia of the beam cross section; $a$ and $l$ are shown in figure 2.

The moment of inertia at any section of the spindle is

$$I = \frac{\pi r^4}{4} \tag{12}$$

where $r$ is the radius of the spindle cross section. As follows from figure 2, $r = kx$ and therefore

$$I = \frac{\pi k^4 x^4}{4} \tag{13}$$

Conversely, the bending moment due to unit force applied at the distance $a$ from the origin of the coordinates (at the tip of the beam) is

$$M = x - a$$

The expressions for $I$ and $M$ are substituted in equation (11) to give

$$\Delta_{11} = \frac{4}{\pi k^4 E} \int_a^b \frac{|x - a|^2}{x^4} dx \tag{11a}$$

By integrating and substituting the limits

$$\Delta_{11} = \frac{4}{\pi k^4 E} \left[\frac{1}{3a} + \frac{a}{b^2} - \frac{1}{b} - \frac{a^2}{3b^3}\right] \tag{14}$$

By denoting the largest diameter of the spindle by $R$ and the smallest diameter by $r_0$, $k$, $a$, and $b$ are determined. In accordance with figure 2

$$k = \frac{R - r_0}{l} \tag{15}$$

$$a = \frac{r_0 l}{R - r_0} \tag{16}$$

$$b = l + \frac{r_0 l}{R - r_0} = l \frac{R}{R - r_0} \tag{17}$$

Finally

$$\frac{1}{\delta_{11}} = \Delta_{11} = \frac{4l^3}{\pi E\left|R - r_0\right|^3}\left[\frac{r_0}{R^2} + \frac{1}{3r_0} - \frac{1}{R} - \frac{r_0^2}{3R^3}\right] \tag{18}$$

The obtained result is substituted in the formula for $\omega_{cr}$ to give

$$\omega_{cr} = \sqrt{\frac{h^2 \pi E\left|R - r_0\right|^3}{\left|A - C\right| \left|\frac{r_0}{R^2} + \frac{1}{3r_0} - \frac{1}{R} - \frac{r_0^2}{3R^3}\right| 4l^3}} \tag{19}$$

Because ultracentrifuges operate at velocities higher than the critical, it is impossible to ignore the passing of the rotor through the critical point.

Factors that affect the passage of the centrifuge rotor through the critical velocity are to be considered.

Two methods exist by which the passage of the ultracentrifuge rotors through the critical velocities may be affected. The first method provides a rapid start that excludes an inadmissible increase in energy of the forced vibrations and may be applied to small centrifuges of various types. The second method restricts the amount of bending of the spindle when passing through the critical point by means of rings formed by suitable plates. The second method is the one applied most often and will be subsequently considered with the aid of a method proposed by Kapitsa (reference 2).

As has been shown by Kapitsa, the critical point separates two qualitatively different types of rotation. The transition from one motion to the other for the ultracentrifuge is discussed.

In the first period of the motion, the gyroscopic couple balanced by the elastic force of the spindle is determined by the left side of equation (8).

For a certain value of the angular velocity $\omega_1$, the head of the rotor touches the restricting surface. The gap between the head and the ring is denoted by $e$. The angle of inclination of the rotor axis corresponding to the contact of the restricting ring will be equal to $\beta_1 = e/h_3$, where $h_3$ is the distance from the lower support to the ring.

The value of $\beta_1$ is substituted in equation (8) and is solved for $\omega_1{}^2$ by replacing $-h^2\delta_{11}/C-A$ with the value $\omega_{cr}{}^2$. Thus

$$\omega_1 = \omega_{cr} \frac{e}{h_3 \beta_0 + e} \tag{20}$$

When the velocity of the rotor reaches the value $\omega_1$, the deflection of the shaft ceases to increase because the restricting ring prevents it.

After contact and with increase in angular velocity, the head of the rotor presses against the restricting ring through the centrifugal force. At first the head slides along the surface. When the friction force between the head and the ring reaches a certain value, however, the rotor starts to roll along the ring in the opposite direction and the spindle immediately straightens out.

The value of $\omega_2$ corresponding to the straightening out of the spindle is determined.

The contact of the rotor with the ring changes the value of the moment of the external forces determined by equation (1). The moment of the reaction force of the ring $Qh_3$ is added to the moment of the elastic force of the bending spindle.

The angle between the plane of action of the gyroscopic moment and the bending plane including the point of contact is denoted by $\gamma$; the moment acting in this case is determined by the equation

$$\omega^2 \left| C - A \right| \left| \beta_1 + \beta_0 \cos \gamma \right|$$

By denoting as $Q$ the reaction force of the ring, the following equation may be written:

$$\omega^2 \left| C - A \right| \left| \beta_1 + \beta_0 \cos \gamma \right| = - \delta_{11} h^2 \beta - Qh_3 \tag{21}$$

The sliding of the head of the rotor along the ring is accompanied by the appearance of the tangential force $\mu Q$, where $\mu$ is the coefficient of friction. The condition for the existence of motion of the first kind must be

$$\frac{\beta_0 \omega^2 \left| A - C \right| \sin \gamma}{h_3} > Q\mu \tag{22}$$

From equations (21) and (22), the following equation is obtained by the Kapitsa method (reference 2):

$$\frac{1}{\mu} \sin \gamma - \cos \gamma > \frac{\beta_1}{\beta_0} \frac{\omega_2{}^2 - \omega_{cr}}{\omega_2{}^2} \tag{23}$$

$$\frac{\omega_2{}^2}{\omega_{cr}{}^2} = \frac{\beta_1}{\beta_1 - \beta_0 \sqrt{1 + \frac{1}{\mu^2}}} \tag{23a}$$

where $\omega_2$ is the angular velocity corresponding to the straightening of the spindle. The condition of the passage through the critical velocity is obtained in the form

$$\frac{\beta_1}{\beta_0} > \sqrt{1 + \frac{1}{\mu^2}} \tag{24}$$

By denoting the gap between the head of the rotor and the ring as $e$

$$\frac{e}{h_3\beta_0} > \sqrt{1 + \frac{1}{\mu^2}} \tag{25}$$

The preceding inequality shows that the ultracentrifuge rotor cannot pass through the critical velocity if the space between the head and the ring is small, if the coefficient of friction between them is small, or finally, if the unbalance of the rotor is large.

The increase in the gap $e$, however, is limited by the strength of the spindle.

The maximum permissible value of $e$ is determined. The displacement of the end of the rotor spindle under the action of a unit force is $\Delta_{11}$. The stress arising at the place where the spindle is stopped, in the case where the force $x$ is applied to the end of the spindle, is

$$\sigma = xl/W$$

where $W$ is the moment of resistance of the section and the deflection is equal to $\Delta_{11}x$. By giving a value of the deflection equal to $e$ corresponding to the occurrence in the critical section of a stress equal to the permissible stress $\sigma_z$

$$e = \Delta_{11}x = \frac{\Delta_{11}\ \sigma_z W}{l} \tag{26}$$

In accordance with equation (18)

$$e = \frac{l^2\ R^4\ \sigma_z}{E\left|R - r_0\right|^3}\left|\frac{r_0}{R^2} + \frac{1}{3r_0} - \frac{1}{R} - \frac{r_0}{3R^2}\right| \tag{27}$$

or

$$e = \frac{l^2\ \sigma_z}{E\left|R - r_0\right|^3}\left|r_0\ R^2 + \frac{R^4}{3r_0} - R^3 - \frac{r_0{}^2 R}{3}\right| \tag{28}$$

The pressure at the upper bearing and at the instant of the passage of the rotor through the critical velocity is determined.

In equation (26), the pressure on the bearing is evidently equal to the magnitude $x$ or

$$x = \frac{\sigma_z W}{l} \tag{29}$$

After passing through the critical point, the spindle remains bent by an amount practically equal to $\beta_0 h$. The force due to the bending transmitted on the bearing is equal to

$$x_1 = \frac{\delta_{11}\ \beta_0}{h} \tag{30}$$

or by substituting the value $\delta_{11}$ from equation (10)

$$x_1 = \frac{\omega_{cr}^2 \left|A - C\right| \beta_0}{h^3} \tag{31}$$

## CONCLUSIONS

Particular attention is to be paid to the problem of the passage of an utlracentrifuge rotor through the critical velocity. The difficulties encountered in the practical use of ultracentrifuges are connected with this passage.

The dynamical balance of the rotors has often been neglected, whereas their balance increases in connection with the too frequent removal of the rotors from the body of the centrifuge for cleaning. A result of the increase in the unbalance may be the loss in the ability to pass through the critical velocity and the rapid wear of the bearings (the force acting on the bearing, equation (31)).

The preceding discussion points out the factors that affect the passage of rotors through the critical point, namely, the gap between the head and the ring, the coefficient of friction, and the unbalance. These factors cannot be excluded in the operation and design of ultracentrifuges.

Translated by S. Reiss,
National Advisory Committee
for Aeronautics.

## REFERENCES

1. Nikolai, E. L.: Theoretical Mechanics, III, 161, 1939.

2. Kapitsa, P. L.: Jour. Tech. Phys. (U.S.S.R.), vol. 9, no. 2, 1939.

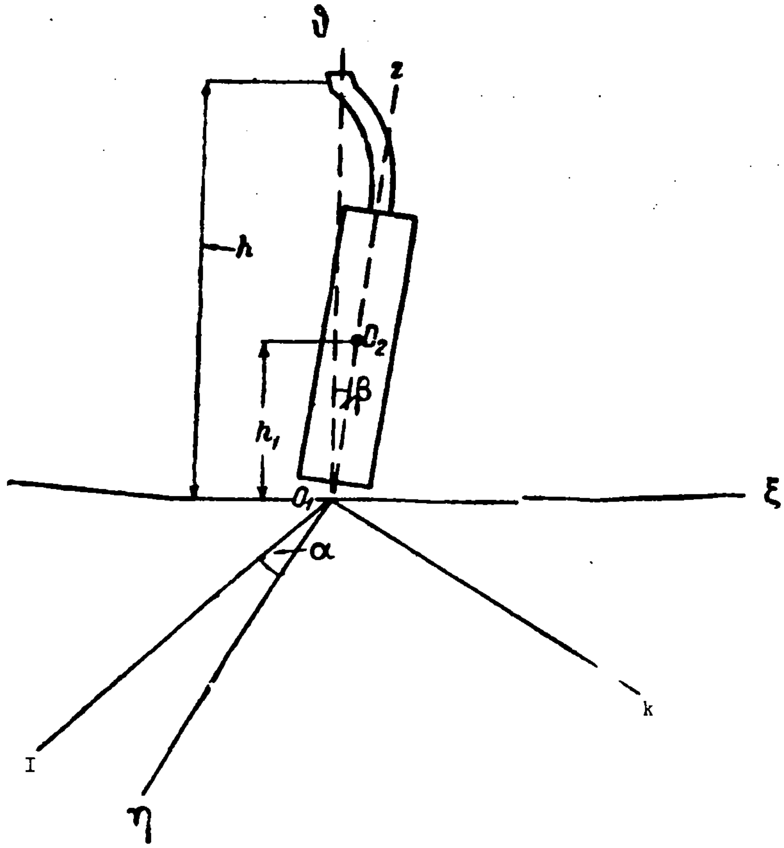


Figure 1.

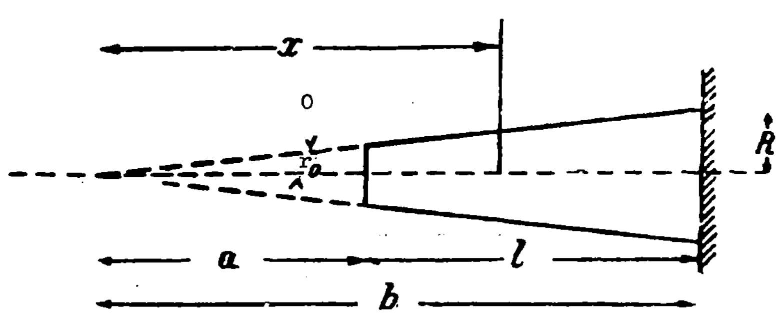


Figure 2.